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Spin 1/2 Fermions in the Unitary Limit

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Abstract

The energy-shift due to the interaction of two particles in a large box is proportional to the free particle scattering phase-shift. This provides an approximation to the effective interaction referred to as the phase-shift approximation. For a many-body fermion system this effective interaction has to be corrected for the Pauli-blocking. It is used to calculate the energy of a spin $\frac{1}{2}$ fermion system as a function of the two-body scattering length a and effective range r_0. In the unitary limit $a \to -\infty, r_0 \to 0$ the energy is $\xi = 0.540$ (in units of the fermi-gas) with pp-ladders. Including also hh-ladders $\xi = 0.572$. A smooth crossover from the BCS to the BEC region is observed.

For comparison a separate calculation is made with a separable interaction obtained from the same phaseshifts (defined by a and r_0) by inverse scattering. The energy calculated with this interaction in the ladder approximation agrees with the previous except in the unitary limit where this method as expected breaks down.

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1 Introduction

The properties of a dilute fermigas with large scattering length has become of considerable theoretical as well as experimental interest. The inner crust of neutron stars may consist of a low density neutron gas[1]. Taking advantage of Feshbach resonances it is now also possible to magnetically tune the atomic scattering lengths,e.g.[2]. Increasing the scattering length of fermions from $-$ to $+\infty$ resulting in bound boson systems to explore the crossover from BCS to BEC has been reported by several groups.

A theoretically related problem proposed by George Bertsch is that of the energy of a dilute system of spin 1/2 fermions interacting via a zero-range, infinite scattering length interaction.[3] For such a system one would expect the existence of a constant ξ being a function only of fundamental constants such that the total energy $E = \xi E_{FG}$ where E_{FG} is the uncorrelated Fermi-gas energy.

Several numerical methods have been used to determine ξ. Most recent of these appears to be the Monte Carlo calculations of Carlson et al.[4] giving $\xi = 0.44 \pm 0.01$. Other authors report values of $\xi = 0.326$. [5, 6]

This paper is a report on results obtained using two separate but similar methods, both involving a ladder summation by the Brueckner method including the Pauli-operator but neglecting the mean field dispersion correction. In the first use is made of the well known fact that the interaction is separable for large scattering lengths, which is the case of interest here. The separable potential is found by inverse scattering techniques from a given set of phase-shifts at low energies defined by a scattering length a and an effective range r_0. This technique has been used and described in previous work [7, 8]. A brief derivation is found in Section II. In the limit $a \to -\infty$ and $r_0 \to 0$ the phase-shifts become constant ($= \pi/2$) and the separable potential will have an infinte range in momentum-space and this method breaks down.

This necessitates a different approach. The phase-shift approximation [10, 11, 12] used in early Brueckner calculations is extended to include the Pauli operator. This method was used in some earlier work with the assumption that the phase-shifts are 'separable'.[13] Details are found below in Section III. Results of the numerical work are shown in Section IV. In Section V these results are compared with some previous works and Section VI contains a summary and some further comments.

2 Separable Interaction

The use of a separable interaction in Nuclear Physics problems has a long history. It seems however that the first consistent calculation using inverse scattering techniques to construct a separable NN potential with an application to the nuclear matter problem was that reported in ref [7]. A close agreement with calculations using the meson-theoretical potentials of Machleidt was found. Subsequent use was shown in ref[8, 9] relating to V_{low-k} etc. In the latter of these two last references the dispersion corrections and its relation to saturation was of primary interest. It is well known that for a two-particle system with a bound state at or close to zero energy the interaction can be represented by a separable potential. The method described below in which this

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separable potential is obtained by inverse scattering is therefore suitable when considering the problem at hand, large scattering lengths. One of the main problems in an inverse scattering calculation is the change in sign of the phasehift as a function of relative momentum. In the present case of low density and a low momentum interaction this is not an issue. In the simplest representation one can use a rank 1 potential

$$V(k,p) = \lambda v(k) v(p) \tag{1}$$

with $\lambda = \pm 1$. Inverse scattering then yields, if no bound states occur (e.g. ref [7, 14])

$$v^2(k) = -\lambda \frac{(4\pi)^2}{k} sin\delta(k) |D(k^2)| \tag{2}$$

where

$$D(k^2) = exp\left[\frac{2}{\pi}\mathcal{P}\int_0^\Lambda \frac{k'\delta(k')}{k^2 - k'^2} dk'\right] \tag{3}$$

where $\mathcal{P}$ denotes the principal value and where $\delta(k)$ is the phaseshift. Λ provides a cut-off in momentum-space. In the spirit of V_{low-k} the result should not involve high momenta. The effect of the cut-off will be explored further here. This inverse scattering solution is of course not unique but the calculation of the energy in the limit $a \to -\infty$ should not depend on the details of the potential.

The Brueckner G-matrix [2] is given by

$$G(k,p,P) = \frac{\lambda v(k) v(p)}{1 - \lambda I(P,k)} \tag{4}$$

with

$$I(P,k) = \frac{1}{(2\pi)^3}\int v^2(k') \frac{Q(P,k')}{k^2 - k'^2} k'^2 dk' \tag{5}$$

where P is the center of mass momentum and Q is the Pauli-operator in the angle-averaged approximation. The dipersion correction (the mean field) is not included in the calculations below and the energy in the denominator is only the kinetic.

The total energy is obtained in the usual fashion:

$$E_{tot}/A = \frac{3}{10}\frac{\hbar^2}{2m} + \frac{1}{(2\pi)^6}\int G(k,k,P) F(k,P) d\mathbf{k} d\mathbf{P} \tag{6}$$

with $F(k,P)$ the probability distribution of k and P in the Fermi-sea.

3 Phase-shift Method

In scattering theory one defines a T-matrix by

$$T^{\pm}(\mathbf{k},\mathbf{p},\omega) = V(\mathbf{k},\mathbf{p}) + \frac{1}{(2\pi)^3}\int d\mathbf{k}' V(\mathbf{k},\mathbf{k}') \frac{1}{\omega - k'^2 \pm i\eta} T^{\pm}(\mathbf{k}',\mathbf{p},\omega) \tag{7}$$

with on-shell diagonal elements

$$T(\mathbf{k},\mathbf{k},\omega = k^2) = \frac{e^{i\delta(k)} sin\delta(k)}{k}$$

Related to the T-matrix is the Reactance matrix R defined by [3]

$$R(\mathbf{k},\mathbf{p},\omega) = V(\mathbf{k},\mathbf{p}) + \frac{1}{(2\pi)^3}\mathcal{P}\int d\mathbf{k}' V(\mathbf{k},\mathbf{k}') \frac{1}{\omega - k'^2} R(\mathbf{k}',\mathbf{p},\omega) \tag{8}$$

It is related to the T-matrix by the operator equation

$$\mathbf{T} = -\mathbf{R}(\mathbf{1} - i\mathbf{R})^{-1} \tag{9}$$

[2] By Brueckner and in most past papers by the present author refered to as the K-matrix

[3] In the literature also referred to as the K-matrx

The diagonal elements of R are

$$R(\mathbf{k},\mathbf{k},\omega=k^2)=\frac{tan\delta(k)}{k} \tag{10}$$

The solution of eq. (8) is with the separable potential similar to that of eq. (4) but with $I(\omega)$ given by

$$I(\omega)=\frac{1}{(2\pi)^3}\mathcal{P}\int\frac{v^2(k')}{\omega-k'^2}k'^2dk' \tag{11}$$

In the early days of Brueckner's development of the many -body theory bearing his name the R-matrix (i.e. $tan\delta/k$) was used as an approximation of the effective interaction.[15, 16] It was subsequently shown that a more appropriate approximation is δ/k [10, 11, 12], and this is referred to as the *phase-shift approximation.* This is in agreement with considering the shift in energy due to the interaction of two particles in a large spherical enclosure satifying the appropriate boundary conditions.[11, 17, 18, 19] The important point here is that the R-matrix refers to scattering in an open infinite system with a continuous spectrum. But the problem at hand, the binding energy of a system in a *large* but finite container presents different boundary conditions with a discrete spectrum. The transition to the continuum as the size of the container goes to ∞ is not trivial. This point has also been forwarded by Anderson [20].

The phase-shift approximation was in fact used already a long time ago to calculate the energy of a low density neutron gas [21, 22]. In this approximation the potential energy per particle is given by

$$\frac{P.E.}{N}=-\frac{8\hbar^2}{\pi m}\int_0^{k_f}k\delta(k)dk\left(1-\frac{3}{2}\frac{k}{k_f}+\frac{1}{2}\frac{k^3}{k_f^3}\right). \tag{12}$$

$F(k,P)$ in eq. (6) is here independent of P. In the limit $a\to-\infty$ and $r_0\to 0$ one finds $\delta\to\frac{\pi}{2}$ and adding the kinetic energy to eq. (12) yields a total energy $E=\xi E_{FG}$ with $\xi=-1/3$, independent of k_f. Estimates [22] showed that the effect of the Pauli-operator (neglected in eq. 12) at low densities and with the $^1S_0$ scattering length and effective range, is to reduce the potential energy by a factor of ~ 0.5. Using this same estimate for $\delta=\frac{\pi}{2}$ one finds $\xi\sim+1/3$ suggesting the importance of including the Pauli-operator in this case. An accurate treatment of the Pauli-operator together with the phase-shift approximation is called for.

In order to relate to the Brueckner G-matrix a brief review of the phase-shift approximation derived in refs [10, 11, 12] will first be presented. Consider a spherical box of radius R_0. It is important that the box is spherical rather than cubical to avoid complications [10]. To find the energy-shift due to the two-particle interaction in this box Riesenfeld and Watson uses a selfconsistency condition defining a matrix $\mathcal{R}$ by a Brillouin-Wigner relation [4]

$$\mathcal{R}(k,p)=V(k,p)+\sum_{k'\neq k}V(k,k')\frac{1}{e(k)-e(k')+\Delta E(k)}\mathcal{R}(k',p) \tag{13}$$

with

$$\Delta E(k)=\mathcal{R}(k,k) \tag{14}$$

being the level shift for two particles interacting with relative momentum k. This matrix obviously differs from the Reactance-matrix R defined by eq. (8) by the $\Delta E(k)$ in the energy-denominator. Another very important difference is that the energies here are discrete, being those for the large spherical box while those defining the Reactance matrix are continuous scattering states and this important issue is returned to below.

The difference between the two matrices can be expressed by the relation

$$\mathcal{R}(k,p)=R(k,p)+\sum R(k,k')\left[\frac{1}{e(k)-e(k')+\Delta E(k)}-\frac{\mathcal{P}}{e(k)-e(k')}\right]\mathcal{R}(k',p) \tag{15}$$

The reactance matrix was defined in eq. (8) with an integral over intermediate states instead of a sum as used above. This makes in general no difference with the volume of the box $\mathcal{V}\to\infty$. It is however important here where the correct transition from a discrete to a continous spectrum is at issue as e.g. emphasized in ref.[10].

The level separation dE in the unperturbed energy spectrum as well as the shift $\Delta E(k)$ is of the order of $\mathcal{V}^{-1}$. In an early paper trying to justify Brueckner's choice of $tan\delta/k$ as an effective interaction, Reifman et al [23] argued that the levels surrounding the level perturbed by $\Delta E(K)$ approach this perturbed level *symmetrically* . This would imply that $\Delta E(k)$ in eq. (15) can be ignored so that $\mathcal{R}\to R$ as $dE\to 0$. An important point for

[4] They denote this matrix by R above used for the Reactance matrix

the following results is that this small shift cannot be ignored. The integration across the pole in the energy denominator has to be done with care as originally stressed by Fukuda and Newton[11].

This is due to an important *asymmetry* rather than the earlier assumed *symmetry* . This is seen as follows. The difference betwen the two propagators in eq. (15) can be written as

$$\frac{1}{e(k)-e(k')+\Delta E(k)}-\frac{\mathcal{P}}{e(k)-e(k')}=-\Delta E(k)[(e(k)-e(k')+\Delta E(k))(e(k)-e(k')]^{-1} \tag{16}$$

It is now argued [12] that this correction term to the Principal value term is important only when summing over states around $k' \sim k$.

The discrete spectrum level density in momentum space is

$$\frac{dn}{dk}=\frac{R_0}{\pi}$$

so that one can write

$$e(k)-e(k')=kdk=\frac{\pi k}{R_0}(n'-n)$$

With

$$q=n'-n$$

and

$$\sigma=-\frac{R_0}{\pi k}\Delta E(k)$$

one finds

$$\sum_{k'\neq k}-\Delta E(k)[(e(k)-e(k')+\Delta E(k))(e(k)-e(k')]^{-1}=-\Delta E(k)\left[\frac{R_0}{\pi k}\right]^2\sum_{q\neq 0}\frac{1}{q(\sigma+q)} \tag{17}$$

The sum in this equation can be done analytically and yields[12]

$$\frac{\pi}{\sigma}\left[\frac{1}{\pi\sigma}-\frac{1}{tan\pi\sigma}\right]$$

The asymmetry alluded to above is a consequence of the nonzero σ in the expressions above.

Because only the states close to k contribute in the summation above one then obtains

$$\mathcal{R}(k,p)=R(k,p)+R(k,k)\gamma(k)\mathcal{R}(k,p) \tag{18}$$

with

$$\gamma(k)=k\left[\frac{1}{\delta(k)}-\frac{1}{tan\delta(k)}\right] \tag{19}$$

With eqs (14) and (10) one then finds

$$\mathcal{R}(k,k)=\delta(k)/k. \tag{20}$$

It has thus been found that if neglecting the Pauli-operator and the dispersion-effect in the Brueckner G-matrix and using the proper limiting procedure the result is eq. (20). It is referred to as the *phase-shift approximation* . When tested against the full G-matrix in calculations of the total energy it breaks down especially for large densities.

It should be noted that the same limiting procedure should in principle be applied when solving eq. (4) for the G-matrix. There the appearance of the Q-operator which excludes the $k'=k$ states in the summations above makes this however irrelevant[25], except perhaps at the Fermi-surface.

The following relation will now be used to extend the phase-shift-approximation to include the Pauli-operator. As above the dispersion-effect will be neglected at this point. [5]

$$G(k,k,P)=\mathcal{R}(k,k)+\frac{1}{(2\pi)^3}\int d\mathbf{k}'\mathcal{R}(k,k')\left[\frac{Q(P,k')}{k^2-k'^2}-\frac{1}{k^2-k'^2+\Delta E(k)}\right]G(k',k,P) \tag{21}$$

[5]It was included in ref. [13]

The results above shows that

$$\frac{1}{k^2-k'^2+\Delta E(k)} = \frac{\mathcal{P}}{k^2-k'^2}+\gamma(k) \tag{22}$$

To solve eq. (21) for G the separable assumption mentioned above will be made. The 'phase-shift matrix' $\mathcal{R}$ is assumed to be separable with diagonal elements being $\delta(k)$ as already shown. One then finds similar to eq. (4)

$$G(k,k,P) = \frac{\delta(k)}{1-I(P,k)} \tag{23}$$

with I now given by

$$I(P,k) = \frac{1}{(2\pi)^3}\int k'dk'\delta(k')\left[\frac{Q(P,k')}{k^2-k'^2}-\frac{\mathcal{P}}{k^2-k'^2}+\gamma(k)\right] \tag{24}$$

Using the separable assumption the relation (22) will also be used below to calculate the diagonal of the $\mathcal{R}$-matrix, eq. (20), from eq. (23) with $G \to \mathcal{R}$ and with

$$I(k) = \frac{1}{(2\pi)^3}\int k'dk'\delta(k')\left[\frac{\mathcal{P}}{k^2-k'^2}+\gamma(k)\right] \tag{25}$$

This result together with eq. (20) is used as a consistency check of the calculations as the phase-shifts $\delta(k)$ obtained from the diagonal of $\mathcal{R}$ should agree with the input $\delta(k)$.

4 Numerical Results

Results shown below are in the Brueckner *pp* ladder-approximation but in one figure the effect of including *hh* ladders is also shown. Standard techniques are used solving for the matrices. The angle-averaging is used for the Pauli-operator but when using the separable phase-shift method an exact treatment is also used but with no signifcant difference. The momentum mesh in the integrations were typically $0.01fm^{-1}$ for $k_f > 0.3fm^{-1}$ and 0.005 otherwise. The total energy is expressed in terms of ξ with $E_{total} = \xi E_{FG}$ with E_{FG} being the neutron-gas (non-interacting) kinetic energy. As already mentioned above the dispesion-correction i.e. a mean field is not included in the definition of the G-matrices.

4.1 Comparing the potential and phase-shift methods

It is of interest to compare the two methods of Sects II and III numerically. Tables below show results using the Arndt [24] phases for $E > 1MeV$ supplemented by phases from the $^1S_0$ n-n scattering length $a = -18.5$ and the effective range $r_0 = 2.85$ for $E < 1MEV$.

Table I shows the result using the separable potential calculated by inverse scattering from these phase-shifts as a numerical input. The obtained potential was tested for accuracy by calculating the diagonal of the R-matrix using eq. (10) and the $\mathcal{R}$-matrix using eq.(20).

Table II shows results using the same phase-shifts as in Table I but using the phase-shift method described in Section III. The second column, $(P.E./N)_0$, shows the phase-shift approximation given by eq. (12). This implies that the Pauli-correction is not included in this column. In the fourth column, $P.E./N$, the effective interaction G is calculated from eq. (23) that includes the Pauli-correction. In agreement with ref.[22] one finds that this correction reduces the potential energy by a factor of approximately 1/2.

Tables I and II both use the same input phases so ideally the two results should agree and do so within expectancy.

The two methods are compared further in Fig. 1 showing the energy ξ as a function of $\log(-ak_f)$. One finds here that the two methods agree very well in the limit of small and large values of ak_f but less so in the intermediate region where $ak_f \sim r_0k_f$, the region of pairing instability. At this point it is of interest to observe the very different regions of 'intermediate' state summations in the two cases. In the potential method the summation is over *un* -occupied states while in the phase-shift method the summation is over occupied states only and involving a principal value integration.

It should also be pointed out that both methods use the same set of phase-shifts which fixes zero-order diagonal on-shell matrix-elements. The calculation of the in-medium effective interaction requires off-diagonal elements that are obtained by similar but not identical *prescriptions* , i.e. assumption of separability.

TABLE I

SEPARABLE POTENTIAL
$(a = -18.5 \quad r_0 = 2.85)$

k_f	ρ	K.E./N	P.E./N	E/N	ξ
0.1	0.00003	0.124	−0.042	0.082	0.661
0.2	0.0003	0.498	−0.226	0.271	0.544
0.3	0.0009	1.120	−0.539	0.581	0.518
0.4	0.0022	1.991	−0.971	1.020	0.512
0.5	0.0042	3.110	−1.556	1.554	0.500
0.6	0.0073	4.479	−2.258	2.221	0.496
0.7	0.0116	6.096	−3.087	3.009	0.494
0.8	0.0173	7.962	−4.029	3.933	0.494
0.9	0.0246	10.077	−5.103	4.973	0.494
1.0	0.0338	12.441	−6.311	6.130	0.493
1.1	0.0450	15.054	−7.649	7.405	0.492
1.2	0.0584	17.915	−9.135	8.780	0.490

TABLE II

SEPARABLE PHASES
$(a = -18.5 \quad r_0 = 2.85)$

k_f	$(\text{P.E./N})_0$	ξ_0	P.E./N	E/N	ξ
0.1	−0.066	0.467	−0.036	0.088	0.707
0.2	−0.391	0.214	−0.182	0.315	0.634
0.3	−1.016	0.092	−0.467	0.653	0.583
0.4	−1.871	0.060	−0.880	1.110	0.557
0.5	−2.947	0.053	−1.431	1.679	0.540
0.6	−4.205	0.061	−2.120	2.359	0.527
0.7	−5.631	0.076	−2.949	3.146	0.516
0.8	−7.180	0.098	−3.906	4.056	0.509
0.9	−8.857	0.121	−5.074	5.075	0.504
1.0	−10.610	0.147	−6.216	6.225	0.500
1.1	−12.454	0.173	−7.573	7.479	0.497
1.2	−14.381	0.197	−9.081	8.834	0.493

The lowest broken curve in Fig. 1 shows results in the phase-shift approximation showing the importance of the Pauli-correction except at low and high densities. The agreement at low density is expected while the agreement at high density is somewhat unexpected but is because of the cutoff of the phases by the finite r_0 which cuts off the Pauli-correction at the high densities. The phase-shift approximation breaks down however when $r_0 = 0$ giving energy $\xi = -1/3$ [6] while the Pauli-correction is large in this case.

The results above showing agreement of the two methods are with $r_0 > 0$. In the limit of the scattering length $a \to \pm\infty$ and $r_0 \to 0$ the potential is singular and the potential method breaks down as shown below in Fig. 4. This does however not present a problem with the phase-shift method with $\delta(k) \to \pi/2$ in the same limit. Because it has not been well documented or used in the literature.(see however ref. ([13])) it seemed important to compare this method with the separable potential method above. The agreement gives confidence in applying it to the problem at hand in the limit of large scattering length and small effective range, the unitary limit.

[6] See text after eq. (12).

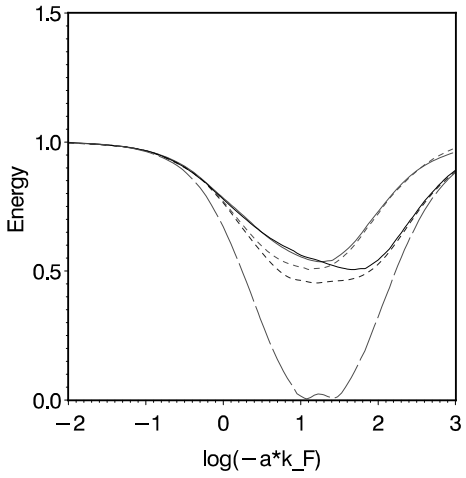

Figure 1: The energy ξ, in units of E_{FG}. The two upper broken lines show results using the separable potential method while the full lines are with the phase-shift method. The uppermost curve in each of these two sets shows ξ with $a = -18.5$ and $r_0 = 2.85$ and the lower is with $a = -100$, $r_0 = 5$ in units of fm^{-1}. The lowest broken curve is with the phase-shift approximation i.e. without Pauliblocking (eq. (20)) and with a fixed at -100 and $r_0 = 5$.

4.2 Limit of Large Scattering Length

Table III shows results of using the phase-shift method with $a = -1000fm$ and $r_0 = 0$ at several densities, the same as in Tables I and II. One should notice that $|ak_F|$ is large at each of these densities; the unitarity region is reached. The energy ξ, in units of E_{FG}, is constant at $\xi = 0.54$. Comparison with Table II shows somewhat smaller values of ξ at the larger densities when using the $^1S_0$ phases.

TABLE III

SEPARABLE PHASES
($a = -1000 \quad r_0 = 0$)

k_f	K.E./N	P.E./N	E/N	ξ
0.1	0.124	−0.057	0.068	0.546
0.2	0.498	−0.227	0.271	0.544
0.3	1.120	−0.513	0.606	0.541
0.4	1.990	−0.909	1.081	0.543
0.5	3.110	−1.427	1.683	0.541
0.6	4.479	−2.055	2.424	0.541
0.7	6.096	−2.805	3.292	0.540
0.8	7.962	−3.662	4.301	0.540
0.9	10.077	−4.644	5.434	0.539
1.0	12.441	−5.728	6.713	0.540
1.1	15.054	−6.938	8.116	0.539
1.2	17.915	−8.282	9.633	0.538

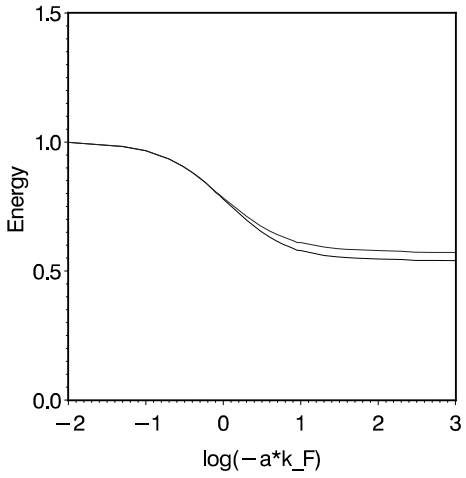

Figure 2: Both lines show the energy ξ, in units of E_{FG}. See text for further information.

Fig. 2 shows results as a function of $\log(-ak_F)$. Of particular interest is of course the unitary regime with $\log(-ak_F) > 2$. The opposite limit is also shown for comparison with Fig. 1. The lower of the two curves is with the Q-operator defined to include only pp-ladders as is customary in Brueckner calculations. The upper line also includes hh-ladders which is customary in Greens function calculations. The Brueckner definition gives $\xi = 0.540$ in the unitary limit as is also the case in the similar calculation shown in Table III. The Greens function calculation gives $\xi = 0.572$. These curves were obtained with $a = -100fm$ and consequently they are functions of k_F. A calculation with $k_F = 1fm^{-1}$ (with scattering length a varying) was also done in the Brueckner case. The result is indistinguishable from the previous, as expected.

The crossover from negative to posive scattering lengths is of particular interest. Fig. 3 shows an enlarged graph of this region. The left part of the graph is the Bose-Einstein condensate (BEC) region with bound pairs of the fermions from the right side. In the unitary region with $1/ak_F \sim 0$ one expects a mixture of fermions and bosons. The formalism used here does not describe the condensation nor the pairing. What is seen in Fig. 3 is instead a continous crossover with a slope that is initially rather constant. The BEC region is accessible in Monte Carlo studies. [26].

5 Comparison with previous work

The first publication relevant for comparing with the present work appears to be that of Baker[6]. He considers an attractive square-well potential with a radius $c \to 0$ and an extrapolation of scattering length $a \to -\infty$. The energy of the system is calculated in a ladder approximation similar to the Brueckner G-matrix as used in the present report. It is however modified to avoid the Emery singularities [27]. The numerical evaluation of the energy using this 'R'-matrix (not to be identified with the Reactance-matrix R above) gives as expected a divergent result for $k_F c \to 0$ present at all scattering lengths. A Padé approximant gives $\xi \sim 0.40$. Baker also provides a series expansion of the ladder sum for $c = 0$. A [2/2] Padé approximant of this sum gives $\xi = 0.568$ while the [1/1] gives $\xi = 0.326$. It seems that his R-matrix method to some extent resembles the methods used in the present paper.

It is interesting to compare his Fig. 2 of ref. [6] with Fig. 4 of this paper. Notice that the potential energy is plotted here as a function of $\hbar^2 k_F^2/M$ for easy comparison with Baker's figure. His Padé approximant gives a nearly straight-line extrapolation as $k_F r_0 \to 0$ contrary to our result. It is however interesting that a straight-line extrapolation in Fig. 4 results in an energy ~ -0.172 close to Baker's -0.180 ($\xi \sim 0.40$), while

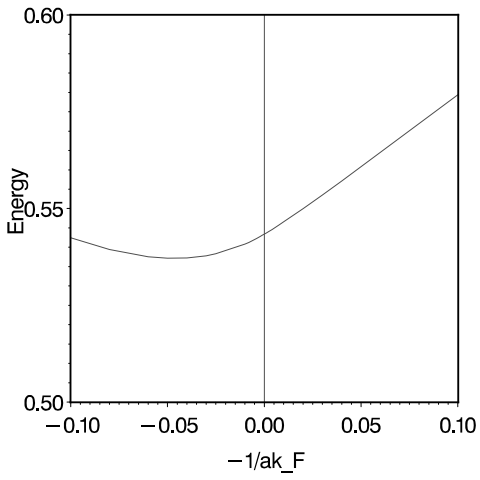

Figure 3: The energy ξ as a function of $-1/ak_F$ to show the crossover from negative to positive scattering lengths.

our result showm in the units of Fig.4 is -0.137. Note that Fig. 4 is a plot as a function of r_0 while Baker's is a function of the range c of his potential. The minimum in the curve at $k_F r_0 \sim 2$ is associated with the term $\gamma(k)$ in eqs (18,19), in particular with the $tan(\delta)$ going rapidly to ∞ as $\delta \to \pi/2$. The short broken curve in Fig. 4 is the result when setting $\gamma = 0$ in eq. (22) and the minimum has disappeared. The difference in the shape of our curves from Baker's for larger $k_F r_0$ is not understood.

The long broken curve is obtained with the separable potential and shows the breakdown of this method for $k_F r_0 \to 0$. This is (not surprisingly) similar to Baker's result. The difference from the phase-shift method is apparent with the large scattering length ($a = -10000 fm$) and $r_0 = 0$ used in this calculation while on the contrary there is close agreement for the values of a and r_0 used in Fig. 1. Heiselberg [5] considers separately the low, intermediate and high density regions. At low density the expansion in $k_F a$ is valid. In this region his Fig. 1 and our Fig. 2 agree (qualitatively). (They differ in units.) At high density his method is non-convergent as expected. In the intermediate densities the $k_F a$ expansion also breaks down. He therefore introduces a ladder summation equation (Galitskii equation) similar to eq. (21) but with a zero order term $\Gamma_0 \sim tan\delta$ while our $\mathcal{R} \sim \delta$. Heiselberg points out that a main problem here is the pairing (Cooper) instability but avoids this problem initially by a momentum average. He finds $\xi = 0.326$.

The Monte Carlo calculations of Carlson et al[4] find a large pairing gap and a $\xi = 0.44 \pm 0.01$ including the pairing contributions. Without a pairing trial function (using a Slater determinant) they obtain $\xi = 0.54$, [7] quite close to our result with pp-ladders shown in Table III and Fig. 2.

6 Summary and discussion

This work addresses the problem of the energy of a zero temperature fermion gas as a function of the scattering length and effective range. The preferred of the two methods which have been applied here is essentially novel in that an expansion of the interaction in terms of the two-body scattering phase-shifts, rather than an interaction potential or scattering amplitude is used to do a Brueckner-type ladder summation. To solve the appropriate integral equation a somewhat drastic assumption was made. The 'phase-shift matrix' was assumed to be separable. This same method was used in an earlier publication[13] with application to nuclear matter. That work was complicated by the change in sign of the phase-shifts, but this is not an issue here. The calculated

[7] I am indebted to K.E. Schmidt to point this out to me.

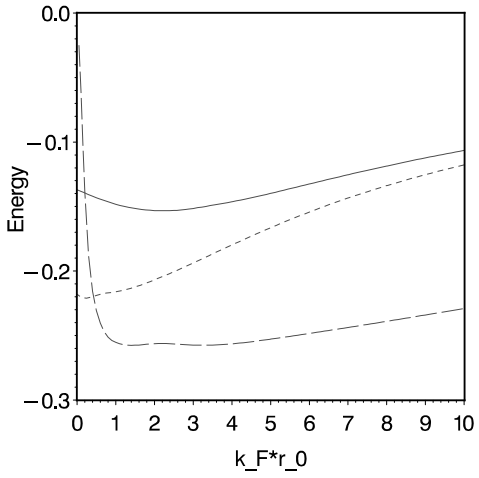

Figure 4: The potential energy in units of $\hbar^2 k_F^2/M$ as a function of $k_F r_0$ with $k_F = 1$. The scattering length is here chosen to be $a = -10000 fm$. The full curve is with the phase-shift method. Notice the slight rise in this curve as $k_F r_0 \to 0$. The short broken curve is with $\gamma(k) = 0$ as discussed in the text. The long dotted curve is with the separable potential showing the breakdown of that method for small r_0 and large a.

binding energy was however surprisingly close to those obtained with realistic potentials for each partial wave except in particular the $^3P_1$ state.

For the problem at hand here the method seems even more promising, especially since conventional perturbative methods break down. The use of separable interactions is for well-known reasons justified for large scattering-lengths, e.g. [28]. In the present problem this may also be justified because the details of the interation is irrelevant as a consequence of the expected universality of the result.

The phase-shift method as used here is an extension of the phase-shift approximation of the Brueckner G-matrix which emerged from the realisation of Fukuda and Newton [11] that the transition from discrete to continous summations requires attention. This is in particular accentuated in the problem of interest here with a large scattering length and the phase-shifts $\delta \to \pi/2$ whence $tan\delta \to \infty$. In this case the phase-shift contains all the information about the interaction and is a natural first-order approximation. The only many-body correction included here is the fermion Pauli-operator. In the present calculations where the mean field is not included in the propagation the summation over inermediate states is then restricted to summation over occupied states only which simplifies the calculation. This also restricts the summation so that practically very small deviations from the diagonal is required.

BCS-pairing has not been included here. It would lead to important corrections. Superfluid gaps have been calculated [5] and are large for $k_F a > 1$. QMC calculations show ξ to decrease by about 0.1 when BCS-correlations are included in the trial wave-function [4]. The hh-ladders were included in calculations shown in Fig. 2. It is known that pairing instability is apt to occur in that case. However none was found in these calculations probably because of the energy spectrum being only kinetic energies; effective mass=1. Instabilities were however found with hh-ladders in a separate calculation where the angle-averaged Pauli-operator was replaced by an exact treatment. The similar pp-ladder calculation did not encounter any instability.

Another important and related situation is the crossover to positive scattering length [26] with formation of bosons [29].

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